

Nonequilibrium critical dynamics of the two-dimensional Ising model quenched from a correlated initial state

László Környei,¹ Michel Pleimling,^{2,*} and Ferenc Igloi^{3,1,†}

¹*Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary*

²*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435, USA*

³*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O.Box 49, Hungary*

(Dated: February 2, 2008)

The universality class, even the order of the transition, of the two-dimensional Ising model depends on the range and the symmetry of the interactions (Onsager model, Baxter-Wu model, Turban model, etc.), but the critical temperature is generally the same due to self-duality. Here we consider a sudden change in the form of the interaction and study the nonequilibrium critical dynamical properties of the nearest-neighbor model. The relaxation of the magnetization and the decay of the autocorrelation function are found to display a power law behavior with characteristic exponents that depend on the universality class of the initial state.

I. INTRODUCTION

The parameters of a physical system are often subject to a sudden change, as for example fast cooling or heating or the switching on or off of an external field. In material science these processes are used to prepare new types of (glassy) states of matter. After such a quench the systems are out of equilibrium and their dynamical properties can be completely different from those known in the equilibrium situation. The phenomena of aging and rejuvenation are typical examples of such glassy dynamics^{1,2}.

In a theoretical approach a simple and often studied example^{3,4,5} is a d -dimensional ferromagnet, which is quenched from its high-temperature (paramagnetic) phase to a temperature T which is below or at the critical temperature T_c . For $T < T_c$ we have phase-ordering kinetics where order grows through domain coarsening⁶. For a critical quench, $T = T_c$, the domains are fractals, and the growth and dynamics involve critical exponents⁷ such as the magnetization scaling dimension $x = \beta/\nu$ (β and ν are the critical exponents of the magnetization and the correlation length, respectively) and the dynamical exponent z .

In this case one often measures the autocorrelation function⁸, $G(t, s) = \langle \sigma(t)\sigma(s) \rangle$, where $\sigma(t)$ is the operator of the magnetization and t and s are the observation time and the waiting time, respectively. Aging is manifested by the fact that $G(t, s)$ is nonstationary, but instead has the scaling form $G(t, s) = t^{-2x/z} \tilde{G}(s/t)$, where the scaling function for small argument behaves as $\lim_{\tau \rightarrow 0} \tilde{G}(\tau) \sim \tau^{(d-x_i-x)/z}$. Here x_i is a new nonequilibrium exponent, the anomalous dimension of the initial magnetization¹⁰. The autocorrelation function for $t \gg s$ behaves as $G(t) \sim t^{-\lambda/z}$, where the autocorrelation exponent satisfies the relation $\lambda = d - x_i + x$. Another common measurement after a critical quench concerns the relaxation of the magnetization⁹, $m(t) = \langle \sigma(t) \rangle$, where the system is initially prepared in a state with a small initial magnetization m_i . Here we have asymptotically $m(t) \sim t^\theta$, where the initial slip exponent is given by $\theta = (x_i - x)/z$.

In critical nonequilibrium dynamics the initial state might contain some kind of correlations. In the case of a perfectly correlated initial state the relaxation process involves only equilibrium exponents. Here we have $m(t) \sim t^{-x/z}$, and, similarly, for $t \gg s$, $G(t) \sim t^{-x/z}$. The dynamic crossover between the ordered and the disordered cases has been the subject of a recent series of papers^{11,12,13}. Another possibility is given by initial states that display quasi-long-range order, i.e. where correlations decay as a power law. This happens for the two-dimensional XY model if both the initial temperature, T_i , and the final temperature of the quench, T , are below the Kosterlitz-Thouless temperature, T_{KT} . If $T_i < T \ll T_{KT}$, we have according to spin-wave theory^{14,15} $G(t) \sim t^{-(x(T)-x(T_i))/z}$, where $x(T)$ is the value of the anomalous dimension at the given temperature. Other recent studies have focused on the d -dimensional spherical model with an initial state of prescribed correlations¹⁶ or on the Ising model with initial states generated through random field effects¹⁷.

In the present paper we consider quenches during which the temperature of the system remains the same, but where the form and the local symmetry of the interactions are changed. Interestingly, the recent progress in the experiments on phase transitions in optical lattices could make this type of investigations possible. We here consider the case of Ising spins on a square lattice with different types of interactions. The phase transition encountered in the Ising model is considered to be the paradigm of an order-disorder transition since the exact solution of Onsager¹⁸ of the standard model with nearest neighbor couplings. The same model, however, with three-spin product interaction for each elementary triangle belongs to a different universality class. According to the exact treatment by Baxter and Wu¹⁹ the (static) critical exponents of this so called Baxter-Wu (BW) model are the same as those of the four-state Potts model²⁰. Still another self-dual model has been introduced by Turban^{21,22,23} and others^{24,25} that has nearest-neighbor interactions in the vertical direction, but n -spin product interactions in the horizontal direction. Of course,

for $n = 2$ we recover the Onsager problem, whereas for $n = 3$, according to symmetry arguments, approximate mappings and numerical investigations, the system belongs to the four-state Potts universality class. For $n \geq 4$ the phase transition is of first order.

In our study we prepare the system in an initial state that is an equilibrium critical state of the BW-model or of the multispin model with $n = 3$ and $n = 4$. In the initial state of the BW-model and that of the $n = 3$ model there are the same type of critical correlations, since the two models belong to the same universality class. On the other hand for the $n = 4$ model we have phase coexistence at the phase transition point between the ordered and the disordered phases. Having prepared the system in this way, we change at time $t = 0$ the form of the interaction and then let the system evolve according to spin-flip dynamics with nearest neighbor interactions. We thereby measure the relaxation of the magnetization and the decay of the autocorrelation function and determine the exponents λ/z and θ .

The structure of the paper is the following. In Sec. II we define the models and describe their critical properties. The results of nonequilibrium relaxation are presented in Secs. III and IV where the initial state corresponds to an equilibrium state of a second order or of a first order transition point, respectively. We discuss our results in the final section.

II. MODELS AND THEIR CRITICAL PROPERTIES

We consider in the following Ising spins, $\sigma_{i,j} = \pm 1$, at the sites of a square lattice with different types of ferromagnetic interactions.

a. Ising model. The standard Ising (or Ising-Lenz) model only contains nearest neighbor ferromagnetic couplings, so that the Hamiltonian is given by

$$\mathcal{H}_I = - \sum_{i,j} J (\sigma_{i,j} \sigma_{i,j+1} + \sigma_{i,j} \sigma_{i+1,j}) . \quad (1)$$

where i and j label the lattice sites, whereas J is the strength of the couplings. The critical point is given by the condition¹⁸

$$\sinh(2J/kT_c) = 1 , \quad (2)$$

which separates a two-fold degenerate ordered phase from a paramagnetic phase. The static critical exponents are known exactly, whereas the dynamical exponent z and the nonequilibrium exponents λ/z and θ are calculated numerically with high precision. The values of these exponents are collected in Table I.

b. Baxter-Wu model. In the Baxter-Wu model we have three-spin product interactions with strength J_{BW} between spins located on elementary triangles, and the

Hamiltonian is given by

$$\mathcal{H}_{BW} = - \sum_{i,j} J_{BW} (\sigma_{i,j} \sigma_{i-1,j} \sigma_{i,j-1} + \sigma_{i,j} \sigma_{i+1,j} \sigma_{i,j+1}) . \quad (3)$$

The ordered phase of the system is four-fold degenerate, with the majority spin orientations in the three equivalent sublattices being given by : $\uparrow, \uparrow, \uparrow$; $\uparrow, \downarrow, \downarrow$; $\downarrow, \uparrow, \downarrow$; and $\downarrow, \downarrow, \uparrow$. According to exact results the critical point of the system is located at¹⁹

$$\sinh(2J_{BW}/kT_c) = 1 , \quad (4)$$

and the static critical exponents are the same as for the four-state Potts model²⁰, but without logarithmic corrections to scaling. Numerical results indicate that universality also holds for the dynamical exponent, z . The nonequilibrium exponents, λ and θ , however, seem to be different, see Table I.

c. Turban model. In this model we have nearest neighbor interactions with strength J_2 in the vertical direction and n -spin product interactions with strength J_n in the horizontal direction, so that the Hamiltonian is given by

$$\mathcal{H}_n = - \sum_{i,j} \left(J_2 \sigma_{i,j} \sigma_{i,j+1} + J_n \prod_{k=0}^{n-1} \sigma_{i+k,j} \right) . \quad (5)$$

The ordered phase of the system is 2^{n-1} -fold degenerate. The model is self-dual^{21,23,24} and the self-dual point is located at

$$\sinh(2J_2/kT_c) \sinh(2J_n/kT_c) = 1 . \quad (6)$$

It is known from numerical studies that a single phase transition takes place in the system, thus the phase transition temperature coincides with the self-duality point. In the following we take $J_2 = J_n$ ($= J = J_{BW}$), so that all models described in this Section have the same critical temperature, see Eqs.(2), (4) and (6). As already mentioned in the introduction the Turban model with $n = 3$ has a continuous phase transition which belongs to the (static) universality class of the four state Potts model^{22,25,26,27,28,29}, even the logarithmic corrections are expected to be of the same form for the two models. The numerical estimates of the dynamical exponent of the two models are somewhat different, although they could be the same within the error of the calculation. The same conclusion holds also for the nonequilibrium exponents, λ and θ , see Table I.

The model for $n \geq 4$ has a first order transition^{30,31}. Detailed numerical studies are available for the $n = 4$ case which exhibits a latent heat of $\Delta/kT_c = 0.146(3)$ and a jump of the magnetization from zero to $m_c = 0.769(6)$. Nonequilibrium relaxation studies of this model have been performed recently³². The autocorrelation function has thereby been found to approach its limiting value, given by the magnetization in the ordered phase

TABLE I: Upper part: static and dynamic critical quantities of the Ising model, the Baxter-Wu model and the Turban model with $n = 3$ and $n = 4$. x : bulk scaling dimension, z : dynamical scaling exponent, λ : autocorrelation exponent, θ : initial slip exponent. For $n = 4$ the phase transition is of first order: ^(a) discontinuity fixed-point value, ^(b) stretched exponential decay. Lower part: nonequilibrium critical exponents of the two-dimensional Ising model starting with an initial state corresponding to the critical state of the Baxter-Wu model and the Turban model with $n = 3$ and $n = 4$.

	x	z	λ/z	θ
Ising	1/8	2.17	$0.74(2)^{33}$	0.187
BW	1/8	$2.29(1)^{34}$	$1.13(6)^{33}$	$-0.186(2)^{34}$
$n = 3$	1/8	$2.3(1)^{35}$	$0.98(2)^{33}$	$-0.03(1)^{35}$
$n = 4$	$0^{(a)}$	$2.05(10)^{32}$	$\infty^{(b)}$	$-1.00(5)^{32}$
BW			0.17(1)	0.18(1)
$n = 3$			0.165(10)	0.18(1)
$n = 4$			0.475(10)	

at the transition point, m_c , through a stretched exponential decay. On the other hand relaxation of the magnetization starting with an uncorrelated initial state with a small magnetization, m_i , has been shown to approach zero with an asymptotic power law time dependence, thus from a nonequilibrium point of view the transition is continuous.

III. RELAXATION FROM SECOND ORDER TRANSITION POINTS

In the following we present results of the relaxation of the critical Ising model where we start from a critical state of the BW model and of the $n = 3$ Turban model. The two models have the same static critical exponents and, interestingly, the decay of the critical correlations is the same as in the critical Ising model. We start to present first the results for the BW model.

A. Relaxation from a BW critical state

In the actual calculations we used finite systems composed of $L \times L$ spins, with L ranging from 60 to 240. We set the temperature to the critical value given in Eq.(4) and then let the system evolve under the Hamiltonian \mathcal{H}_{BW} , see Eq.(3), using the cluster-flip Monte-Carlo algorithm³⁶. After equilibrium is reached the critical BW-states are extracted and sorted by magnetizations. For each magnetization, m_i , we have selected 1000 independent starting states. A typical starting configuration for $m_i = 0$ is shown in the left panel of Fig. 1.

After having selected the initial state, we subject the system to the heat-bath dynamics of the critical Ising model, i.e. the relaxation is performed at the same tem-

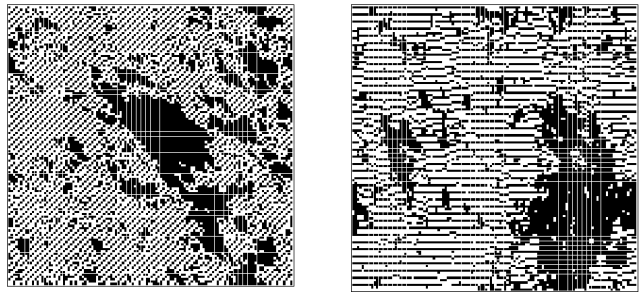


FIG. 1: Typical initial state of the BW-model (left) and the $n = 3$ Turban model (right) with $L = 120$ having a magnetization $m_i = 0$. The initial state is a mixture of the pure phases.

perature as the initialization but the couplings between the spins are changed. For a given starting configuration the relaxation is repeated with typically a few hundred independent sets of random numbers.

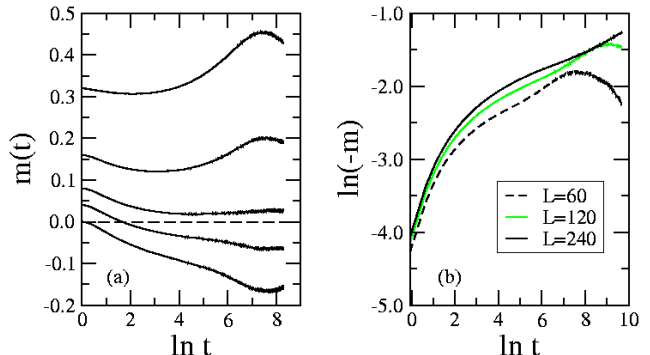


FIG. 2: (Color online) (a) Relaxation of the magnetization starting with a critical BW state with different initial magnetizations m_i (from top to bottom, $m_i = 0.33, 0.16, 0.08, 0.04$, and 0). The data shown have been obtained for $L = 60$. (b) Relaxation of the magnetization starting with a critical BW state with initial magnetization $m_i = 0$ in a log-log scale for different system sizes.

Fig. 2 shows the time dependence of the magnetization for different starting magnetizations m_i . It is seen in Fig. 2a that for small times the magnetization is monotonously decreasing, yielding for $m_i < 0.08$ negative values. The asymptotic dependence of $m(t)$ is studied in more details in Fig. 2b for $m_i = 0$ where we plot the magnetization vs. time in a log-log scale. According to this figure the absolute value of the magnetization has a power-law dependence. For times shorter than some size-dependent time $t_1(L)$ there is an effective exponent, $\theta_1 = 0.13(1)$, which turns to a different value, $\theta = 0.18(1)$, for $t > t_1(L)$. The available time-scale is restricted by the finite size of the system, through $t_m \sim L^z$, z being the dynamical exponent of the critical Ising model, see Table I. The cross-over behavior seen in Fig. 2b is attributed to the structure of the initial state.

As seen in Fig. 1a the initial state can be described as a composition of the four different pure ordered phases of the *BW* model, one with magnetization, $m = 1$, and three others each with magnetization, $m = -1/3$. In the early time steps these pure phases relax and the relaxation of the mixture of the $m = -1/3$ phases has an effective exponent, $\theta_1 = 0.13(1)$. This assumption is in accordance with the effective exponent measured in Fig. 2a for $m_i = 1/3$. Then for $t > t_1(L)$ the remains of the pure $m = -1/3$ phases are dissolved and we are in the true asymptotic regime.

Next we consider the nonequilibrium autocorrelation function, $G(t)$, calculated from a critical *BW* state with vanishing magnetization, $m_i = 0$. The autocorrelation function is thereby defined by

$$G(t) = \frac{1}{L^2} \sum_{i=1}^{L^2} \langle \sigma_i(t) \sigma_i(0) \rangle \quad (7)$$

where we average both over initial states and different realizations of the noise. According to the numerical results shown in Fig. 3a, the autocorrelation function in a finite system of linear size L is well described by the functional form

$$G(t) = At^{-\lambda/z} \exp(-t/\tau_L), \quad (8)$$

where the characteristic time, τ_L , is a monotonously increasing function of L , so that in the thermodynamic limit $G(t)$ has a power law dependence. We tried to fit the measured τ_L in the form $\tau_L \sim L^\zeta$ and for small L the obtained exponent ζ is compatible with the dynamical exponent of the Ising model, as listed in Table I. For larger L , however, when the characteristic time becomes larger than t_1 , the measured ζ is found to decrease below 1.5. A possible explanation of this behavior is that the relevant length scale in the problem, the typical size of the clusters of pure phases, ξ , see Fig. 1a is smaller than L .

In order to deduce the autocorrelation exponent, λ , from the numerical data we have calculated the corrected autocorrelation function, $\tilde{G}(t) = G(t) \exp(t/\tau_L)$, in which we used the estimate for τ_L calculated from Fig. 3a. $\tilde{G}(t)$ is plotted in Fig. 3b in a double-logarithmic scale for different system sizes. Clearly, the curves are seen to approach an asymptotic curve for large L .

Here, as for the relaxation process, one can observe an early time period in which case the effective exponent is $\lambda_1/z \approx 0.14(1)$. In the true asymptotic range the measured autocorrelation exponent is somewhat larger, $\lambda/z = 0.18(1)$. Interestingly, this value is much smaller than the value 0.74 obtained when starting from a fully disordered initial state.

B. Relaxation from an $n = 3$ critical state

For the case of the $n = 3$ Turban model we proceed as for the *BW* case. Using Glauber dynamics, we generate

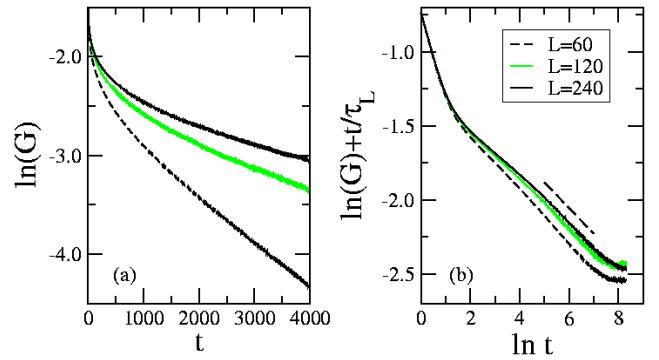


FIG. 3: (Color online) (a) Autocorrelation function starting from a critical *BW* initial state for different system sizes. In the semi-log plot the asymptotic slope is proportional to the inverse characteristic time τ_L . (b) The corrected autocorrelation function, $\tilde{G}(t) = G(t) \exp(t/\tau_L)$, in a double-logarithmic scale for the same system sizes. The slope of the curves corresponds to the exponent λ/z .

equilibrium critical states of the Hamiltonian (5) with three-spin interactions. After equilibrium is reached the initial states are sorted by magnetization (see Fig. 1b for an example). These initial states are then subjected to the heat-bath dynamics of the critical Ising model. The linear sizes of the system used here are the same as for the *BW*-relaxation described in Sec. III A.

Starting with initial states with $m_i = 0$ the magnetization relaxes to negative values. To obtain a qualitative estimate of the initial slip exponent θ we plot in Fig. 4a the absolute magnetization vs. time in a log-log scale. From the asymptotic slope of this curve we obtain $\theta = 0.18(1)$, which is within error bars the value obtained from a *BW* initial state in Sec. III A.

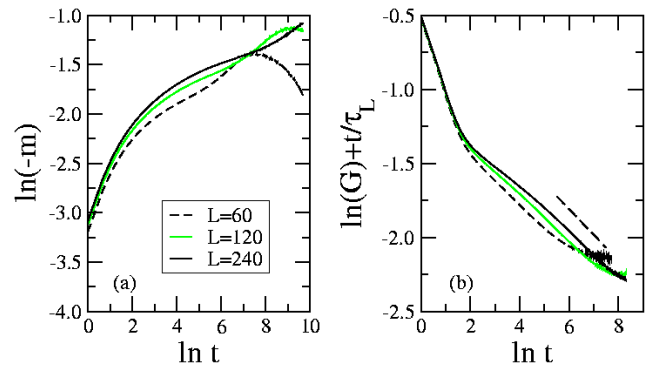


FIG. 4: (Color online) (a) Relaxation of the magnetization starting with a critical $n = 3$ state with initial magnetization $m_i = 0$ in a log-log scale for different system sizes. (b) The corrected autocorrelation function, $\tilde{G}(t) = G(t) \exp(t/\tau_L)$, in a double-logarithmic scale for the $n = 3$ model. The slope of the curves corresponds to the exponent λ/z .

For the autocorrelation function the same ansatz as

given in Eq.(8) works in this case too, and the characteristic time, τ_L , has a similar size dependence as for the BW-model. Having estimated τ_L for each size we have calculated the corrected autocorrelation function, $\tilde{G}(t)$, which is plotted in Fig. 4b in a log-log scale. From the asymptotic slope of these curves we have estimated the autocorrelation exponent as $\lambda/z = 0.165(10)$, which is consistent, within the error of the calculation, with the value obtained from BW initial states in Sec. III A.

IV. RELAXATION FROM A FIRST-ORDER TRANSITION POINT

The Turban model with four-spin product interaction, $n = 4$, has an eight-fold degenerate state in the ordered phase, and this degeneracy is lifted at the first-order transition point at $T = T_c$. The transition point is characterized by the phase coexistence of the paramagnetic phase and the ordered phase, yielding a state that has a completely different structure as the equilibrium critical states of the $n = 3$ model or the BW model considered in the previous section. Since the order-parameter is non-zero even at $T = T_c$, we consider in the following only the nonequilibrium autocorrelation function. Starting with an initial state of the $n = 4$ model at T_c we have used heat-bath dynamics of the critical Ising model.

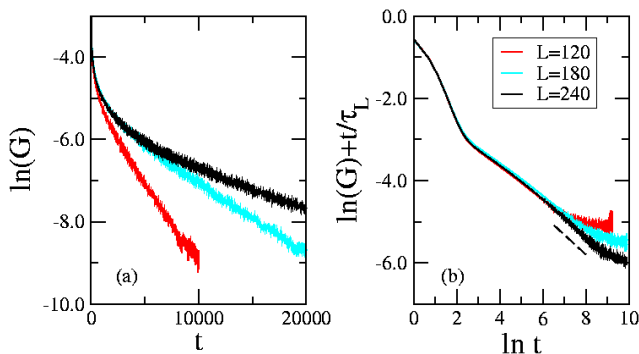


FIG. 5: (Color online) (a) Autocorrelation function starting from a first-order transition equilibrium state of the Turban model with four-spin product interaction for different system sizes. In the semi-log plot the asymptotic slope is proportional to the inverse characteristic time, $\tau(L)$. (b) The corrected autocorrelation function, $\tilde{G}(t) = G(t) \exp(t/\tau_L)$, in a double-logarithmic scale for the same system sizes. The slope of the curves corresponds to the exponent λ/z .

The nonequilibrium autocorrelation function of this system is shown in Fig. 5a. The decay of the autocorrelation is again well described by the same functional form (8) that we have already observed for the two cases discussed before. From the system size dependent characteristic times τ_L we infer a value $z \approx 2.1$ for the dynamical exponent, in good agreement with the literature value 2.17 for the two-dimensional critical Ising model.

The corrected autocorrelation $\tilde{G}(t) = G(t) \exp(t/\tau_L)$ displays a power-law decay as shown in Fig. 5b, but with an exponent $\lambda/z = 0.475(10)$ that is much larger than the value we have obtained when starting from the critical BW or $n = 3$ Turban model. Interestingly, the value of λ/z is compatible with $1/z$, yielding the value $\lambda = 1$ for the autocorrelation exponent.

In the following we present a simple scaling argument which could explain the origin of this numerical finding. First we note that the initial state at $t = 0$, being the state of a system at a first order transition point, is a mixture of clusters of the pure phases and the typical size of the clusters, ξ , is finite and given by the correlation length at the first order transition point. Now, for $t > 0$, we change the form of the interaction and let relax the system during which correlated domains of typical size $\ell(t) \sim t^{1/z}$ are created. In the correlated volume, $\mathcal{V}(t) \sim \ell(t)^d$, due to random fluctuations, a given initial pure phase has an excess volume $\Delta\mathcal{V}(t) \sim \ell(t)^{d/2}$ (compared to the average). Consequently the (corrected) autocorrelation decays as $\tilde{G}(t) \sim \Delta\mathcal{V}(t)/\mathcal{V}(t) \sim t^{-d/2z}$, so that $\lambda = d/2$, in agreement with the numerical results.

V. DISCUSSION

In this paper we have considered a relaxation problem during which the form (and the symmetry) of the interaction between the particles is suddenly changed, whereas the temperature of the system is kept constant. In particular we studied such cases when the system before and after the quench is at a critical temperature, but nevertheless belongs to different universality classes. We have then studied well-known questions in nonequilibrium dynamics, such as the relaxation of the magnetization and the decay of the autocorrelation function. It is known³⁷ that nonequilibrium critical dynamics at time t , after a quench at $t = 0$ from a state with $T_i = \infty$, is analogous to the static critical behavior of a semi-infinite system at a distance y from a free surface located at $y = 0$. The analogous static critical problem to our dynamical problem here is the interface critical behavior at a distance, y , from a straight interface, which separates two coupled semi-infinite critical systems which belong to different universality classes^{39,40}. According to our numerical calculations the nonequilibrium critical behavior in our problem is the result of the interplay and competition between the critical fluctuations of the two systems.

The specific problem we studied is the Ising model on the square lattice, with different types of multispin interactions in the initial states, but with nearest neighbor interactions after the quench. Two initial models, the BW model and the $n = 3$ Turban model, belong to the same static universality class. Interestingly, the magnetization scaling dimension of these models, $x = 1/8$, coincides with that of the normal Ising model. Nevertheless due to the quench, the change of the symmetry of the interaction has a strong effect on the nonequilibrium dynamics.

Our main observation is that nonequilibrium dynamics has the same asymptotic behavior when we start from initial states of different models that are in the same (static) universality class. For the specific problem we considered here, the nonequilibrium exponents are found to satisfy the relation $\theta = \lambda/z$ within the error of the calculation.

The third initial model we considered, the $n = 4$ Turban model, has a first-order phase transition, thus the structure and the topology of the initial state is completely different from the previously discussed cases. Also the asymptotic behavior of the nonequilibrium autocorrelation function is described by a different exponent $\lambda = 1$. This exponent is expected to have the value $\lambda = d/2$ and therefore to depend only on the dimension of the system, but to be universal otherwise, i.e. not dependent

on the type of the initial state, provided it corresponds to a first-order transition point.

Acknowledgments

This work has been supported by the Hungarian National Office of Research and Technology under Grant No. ASEP1111, by a German-Hungarian exchange program (DAAD-MÖB), by the Hungarian National Research Fund under grant No OTKA TO48721, K62588, MO45596. We are indebted to F.Á. Bagaméry for cooperation in the initial period of the project. Some part of the simulations have been done on Virginia Tech's System X.

* Electronic address: Michel.Pleimling@vt.edu

† Electronic address: igloi@szfki.hu

- ¹ N. Kawashima and H. Rieger, *Recent developments in spin glasses*, in: *Frustrated magnetic systems*, ed. H. Diep. World Scientific, Singapore (2005).
- ² M. Henkel, M. Pleimling, and R. Sanctuary (eds.), *Ageing and the glass transition*, Springer Lecture Notes in Physics **716** (Springer, Heidelberg, 2007)
- ³ C. Godrèche and J.M. Luck, J. Phys. Condens. Matter **14**, 1589 (2002).
- ⁴ A. Crisanti and F. Ritort, J. Phys. A **36**, R181 (2003).
- ⁵ P. Calabrese and A. Gambassi, J. Phys. A **38**, R133 (2005).
- ⁶ A.J. Bray, Adv. Phys. **43**, 357 (1994).
- ⁷ see c.f. in: M.E. Fisher, Rep. Prog. Phys. **30**, 615 (1967).
- ⁸ D. Huse, Phys. Rev. **B40**, 304 (1989).
- ⁹ H.K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. **B73**, 539 (1989).
- ¹⁰ H.K. Janssen, *From Phase Transition to Chaos*, eds. G. Györgyi, I. Kondor, L. Sasvári and T. Tél, *Topics in Modern Statistical Physics* (World Scientific, Singapore, 1992).
- ¹¹ P. Calabrese, A. Gambassi, and F. Krzakala, J. Stat. Mech. P06016 (2006).
- ¹² P. Calabrese and A. Gambassi, J. Stat. Mech. P01001 (2007).
- ¹³ R. Paul, A. Gambassi, and G. Schehr, Europhys. Lett. **78**, 10007 (2007).
- ¹⁴ L. Berthier, P.C.W. Holdsworth, and M. Sellitto, J. Phys. A **34**, 1805 (2001).
- ¹⁵ S. Abriet and D. Karevski, Eur. Phys. J. B **37**, 47 (2004).
- ¹⁶ A. Picone and M. Henkel, J. Phys. A **35**, 5575 (2002).
- ¹⁷ L. Környei, M. Pleimling, and F. Iglói, Europhys. Lett. **73**, 197 (2006); L. Környei, J. Phys. Conf. Ser. **40**, 36 (2006).
- ¹⁸ L. Onsager, Phys. Rev. **65**, 117 (1944).
- ¹⁹ R.J. Baxter and F.Y. Wu, Phys. Rev. Lett. **31**, 1294 (1973).
- ²⁰ see c.f. in: R. J. Baxter, *Exactly Solved Models in Statistical*

Mechanics (Academic Press, London, 1982).

- ²¹ L. Turban, J. Phys. Lettres, **43**, L259 (1982).
- ²² L. Turban, J. Phys. C **15**, L65 (1982).
- ²³ J.M. Debierre and L. Turban, J. Phys. A **16**, 3571 (1983).
- ²⁴ C. Gruber, A. Hintermann, and D. Merlini, Group Analysis of Classical Lattice Systems, (Springer, Berlin, New York) 1977, p. 25.
- ²⁵ K.A. Penson, R. Jullien, and P. Pfeuty, Phys. Rev. **B26**, 6334 (1982).
- ²⁶ F. Iglói, D.V. Kapor, M. Skrinjar, and J. Sólyom, J. Phys. A **16**, 4067 (1983).
- ²⁷ H.W.J. Blöte, J. Phys. A **20**, L35 (1987).
- ²⁸ C. Vanderzande and F. Iglói, J. Phys. A **20**, 4539 (1987).
- ²⁹ F. Iglói, J. Phys. A **20**, 5319 (1987).
- ³⁰ F. Iglói, D.V. Kapor, M. Skrinjar, and J. Sólyom J. Phys. A **19**, 1189 (1986).
- ³¹ H.W.J. Blöte, A. Compagner, P.A.M. Cornelissen, A. Hoogland, F. Mallezie, and C. Vanderzande, Physica A **139**, 395 (1986).
- ³² M. Pleimling and F. Iglói, Europhys. Lett. **79**, 56002 (2007).
- ³³ Ch. Chatelain, J. Stat. Mech. P06006 (2004).
- ³⁴ E. Arashiro and J.R. Drugowich de Felício, Phys. Rev. E **67**, 046123 (2003).
- ³⁵ C.S. Simões and J.R. Drugowich de Felício, Mod. Phys. Lett. B **15**, 487 (2001).
- ³⁶ R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. **58**, 86 (1987).
- ³⁷ P. Czerner and U. Ritschel, Physica A **237**, 240 (1997).
- ³⁸ M. Pleimling, J. Phys. A **37**, R79 (2004), and references therein.
- ³⁹ Á. Bagaméry, L. Turban, and F. Iglói, Phys. Rev. B **73**, 144419 (2006).
- ⁴⁰ P. Lajkó, L. Turban, and F. Iglói, preprint arXiv:0709.1322.